2023 Douglas Lectures
Speaker: John McCarthy
(Washington University in St. Louis)

There will be three independent talks. They will all be at the colloquium level, and should be accessible to faculty and graduate students in any field of mathematics. All talks will be 4:00-5:00pm, in Blocker 117.

1. What is a non-commutative function and what is it good for?

(Monday, Nov. 13)

An example of a non-commutative polynomial in 2 variables is

$$p(X,Y) = X^3 + 2X^2Y^3 + 3Y^2X^2Y + 4XY + 5YX + 6.$$

A holomorphic function can be thought of as a generalized polynomial. Similarly, a non-commutative function in d variables is a generalized non-commutative polynomial in d variables. A good choice of domain is sets of d-tuples of n-by-n matrices, where n is allowed to vary over all natural numbers, and one is interested in results that are independent of n.

For example, take the polynomial p above and consider

$$\Sigma := \cup_{n=1}^{\infty}\{(X,Y) : X, Y \text{ are } n\times n \text{ matrices}, \ p(X,Y) = 0\}.$$

Then one can show that if $(X,Y) \in \Sigma$, generically X must commute with Y.

We will explain what exactly a non-commutative function is, and how the theory of non-commutative functions implies the result above.

2. Random commuting matrices

(Tuesday, Nov. 14)

The study of the eigenvalue distribution of random matrices is a well-established field, dating back to the 1920's. It became popular with the work of Wigner and Dyson in the 1950's and 60's, and today is a major field in both probability and theoretical physics.

What can one say about random d-tuples of commuting matrices? What does it even mean?

We will describe one possible approach to defining a random d-tuple of commuting matrices. We shall show that in the Hermitian case, the description of their eigenvalue distribution parallels to some extent the single matrix theory, but in the non-self adjoint case it does not.

3. The Hardy-Weyl algebra and monomial operators

(Wednesday, Nov. 15)

The Weyl algebra is the algebra generated by the operators of differentiation and mulitplication by x. It is much studied in algebra and analysis. What happens if you replace the unbounded differentiation operator by a bounded integral operator, like the Volterra operaor V or the Hardy operator H, where V and H are operators on $L^2[0,1]$ given by

$$\begin{aligned} Vf(x) &= \int_0^x f(t)dt \\ Hf(x) &= \frac{1}{x}\int_0^x f(t)dt? \end{aligned}$$

We shall discuss the algebra generated by multiplication by x and H, which we call the Hardy-Weyl algebra. It is an interesting Banach algebra.

A *monomial operator* is an operator T on $L^2[0,1]$ that has the property that it takes monomials to multiples of other monomials, i.e. $Tx^n = c_n x^{p_n}$. The operators V, H and multiplication by x are all monomial operators. Studying monomial operators one is led to an asymptotic Müntz-Szász theorem.